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Stochastic Portfolio Theory and Stock Market Equilibrium

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I. Introduction

HARRY MARKOWITZ [1952, 1956, 1959] developed a theory of portfolio selection based on the optimization of a quadratic function subject to linear constraints. His work led to the development of a single period equilibrium model, the Sharpe-Lintner capital asset pricing model (CAPM) (see Sharpe [1964], Lintner [1965a, 1965b]), R. C. Merton [1973] extended the CAPM to a continuous time model using lognormal diffusion processes to represent stock price series and showed that the original conclusions continued to hold virtually without change. In this continuous time model it became possible to observe the dynamic interaction between investors' behavior and the behavior of the stocks. In fact, Rosenberg and Ohlson [1976] showed that this interaction led to internal inconsistencies in the continuous time CAPM. In this paper we analyze long term portfolio performance compatible with equilibrium constraints on global portfolio structure.

We present a portfolio theory which is based on Markowitz's theory, but which emphasizes the long term performance of the portfolios in continuous time. The concept of "excess growth" is introduced, a quantity which measures the relative performance of a portfolio compared to that of its component stocks. The equilibrium model we present does not consider questions dealing with optimal strategies for investors, and specifically avoids all normative issues. Rather, it provides constraints on global portfolio structure based on the principle that excess growth is conserved, that the total excess growth in the market at any instant is zero. These constraints generate a distribution of portfolios in the market similar to the energy distribution in thermodynamic equilibrium. This approach provides an alternative to classical supply-demand equilibrium.

We adopt some standard conventions here, but most of the results in the paper will carry over to more general settings. The assumptions we make are:

(1) Each stock price follows a lognormal diffusion process with constant drift and variance parameters. The covariance parameters between stocks are constant. Stock portfolios can be represented as Ito integrals in the various stock price processes.
(2) There are no transaction costs, taxes, or problems with the indivisibility of assets.
(3) The number of shares of each corporation remains constant.
(4) The stocks pay no dividends or other distributions and there is no consumption by investors.
(5) Borrowing of the riskless asset and short sales are permitted without penalty. The riskless rate of return is constant.

In Section II we develop the stock price model and investigate the long term behavior of stocks. In Section III we consider portfolio structure. In Section IV we develop dynamic portfolio theory and the concept of excess growth. In Section V we discuss stock market equilibrium.

II. Stock Price Processes and Their Growth Rates

Since Bachelier [1900], random walk models have been used to represent stock price processes. Although some minor deviations from these models have been noticed (see, e.g., Cootner [1964]), for most purposes they are sufficiently accurate. A current practice is to use a lognormal diffusion process, or its discrete analogue, as a model. Such a process is simply the exponential of a standard random walk. A convenient way to represent a lognormal process is by an Ito integral of the form:

$$dX = \alpha X dt + \alpha X dB \tag{1}$$

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where t is time, B is a standard Brownian motion (Wiener) process with mean 0 and variance t at time t, and α and α are non-negative constants. The constant α is called the expected *rate of return* of X and σ^2 is the *variance*.

It is convenient to let X be the total value of the corporation it represents. That is, there is one share of stock and each shareholder holds a fractional share. This assumption simplifies the exposition without loss of generality.

We shall represent the riskless asset, sometimes referred to as cash, in a similar fashion with $\alpha = r$, the *riskless* rate of return, and $\sigma = 0$. Thus we have

$$d\$ = r\$\, dt \tag{2}$$

where $\$$ represents cash. The riskless asset can usually be treated as a stock, and we shall do so unless otherwise noted. Let us now investigate the long term behavior of these lognormal processes.

Let X(t) be the value of X at time t. Then the expected value of X(t) is

$$EX(t) = [X(0)]\exp(\alpha t) \tag{3}$$

with variance

$$Var\ X(t) = [X(0)]^2 \exp(2\alpha t)[\exp(\sigma^2 t) - 1]. \tag{4}$$

The standard deviation of X(t) is the square root of the variance in equation (4), so we see that while the expected value of X(t) in (3) increases at the rate exp(αt), the standard deviation increases even faster. Hence it is unlikely that the actual value of X(t) will be very near its expected value for large values of t.

Since the expected value of X(t) does not provide an accurate indicator of long term behavior because of the increasing standard deviation, let us approach the problem by integrating the stochastic differential equation in (1). This is simplified by taking the logarithm of X, which, upon application of Ito's lemma (see McKean [1969]), gives us

$$d \log X = (\alpha - \frac{\sigma^2}{2})dt + \sigma dB. \tag{5}$$

This can be integrated directly to give

$$X(t) = X(0)\exp(\alpha t - \frac{\sigma^2 t}{2}) \exp[\sigma B(t)]. \tag{5'}$$

Now, by the strong law of large numbers applied to B(t), we know that for any $\varepsilon > 0$, the inequality

$$-\varepsilon t < B(t) < \varepsilon t \tag{6}$$

will, with probability one, hold for all t greater than some finite value t_o. Therefore, for large enough values of t,

$$\exp(-\varepsilon t) < X(t) \exp(-\gamma t) < \exp(\varepsilon t), \tag{7}$$

where $\gamma = \alpha - \sigma^2/2$ is called the *(compound) growth rate* of X. This shows that for $\sigma^2 > 0$ the sample path of X will not be above its expected value after some finite time t_n, with probability one. In fact, for large values of t, it will be nowhere near its expected value. Therefore, we see that the rate of return of X does not represent the growth of X for large t. In fact, it is clear from the inequalities (7) that the growth rate is the only exponential rate that accurately represents the long term behavior of X. Accordingly, for our analysis the parameters γ and σ^2 are more relevant than the classical parameters α and σ^2.

Now consider the entire stock market composed of many stocks which follow lognormal price processes as in equation (1), but with differing parameters. Each of these stocks has its associated growth rate. A necessary condition for long term equilibrium of the market is that all the growth rates be the same, for if any stock were to have a higher growth rate than the others, that stock would eventually dominate the market and the stocks with lower growth rates would disappear. Hence, all surviving stocks will have the same growth rate and this rate will be shared with the riskless asset. Since equal growth rates are necessary for long term equilibrium, we shall make that assumption when we consider the long term performance of portfolios. When we consider short term phenomena, no such assumption will be made.

For clarity of exposition, the concepts we shall present here are in their simplest form, and natural generalizations can be made to include differing growth rates, alternative stochastic processes, variable numbers of shares of stock, dividends, consumption, and so forth.

III. Portfolio Structure

In this section we discuss portfolio structure and some of the classical results in portfolio theory and stock market equilibrium. Here we shall treat the riskless asset as if it were simply another stock (with zero variance). In fact, there is no need here even to assume the existence of a riskless asset. Let us first discuss the construction of portfolios.

Let the *market* be the family of all stocks and let $X_1,\ldots,X_n$ be the stocks in the market. Two stocks are *equivalent* if they differ by at most a constant factor. This means that for equivalent stocks, the α, σ, and B in (1) are equal. A class of equivalent stocks can be combined and treated as a single stock, and we shall do so.

A *portfolio* Z is a positive valued combination of stocks which satisfies the following conditions. Consider the identity

$$Z = A_1 Z + \ldots + A_n Z. \tag{8}$$

where the A_i are functions depending on time and the stock prices $X_1(t),\ldots,X_n(t)$ such that $\Sigma_I A_I = 1$. Let A_I represent the proportion of Z that is invested in X_I, i.e., $A_I Z$ is invested in X_I. We suppose that Z can be expressed as an Ito integral

$$dZ = \frac{A_1 Z dX_1}{X_1} + \ldots + \frac{A_n Z dx_n}{X_n} \tag{9}$$

or,

$$\frac{dZ}{Z} = \frac{A_1 dX_1}{X_1} + \ldots + \frac{A_n dX_n}{X_n}. \tag{10}$$

A *passive portfolio* is a portfolio in which the number of shares of each stock remains constant. Thus we have

$$Z = v_1 X_1 + \ldots + v_n X_n. \tag{11}$$

where v_I is the (fractional) number of shares of X_I in the portfolio Z. In this case $A_I = v_I X_I / Z$ and

$$dZ = v_1 dX_n + \ldots + v_n dX_n. \tag{12}$$

The *market portfolio* is the passive portfolio

$$M = X_1 + \ldots + X_n, \tag{13}$$

that is, the total value of all stocks in the market. In this case $A_I = X_I/M$ and

$$dM = dX_1 + \ldots + dX_n. \tag{14}$$

A *balanced portfolio* is a portfolio of the form (9) where all the proportions A_I are constants. In this case we have constants π_I, $\Sigma_I \pi_I = 1$, where π_I is the fixed proportion of Z invested in X_I and (10) becomes

$$\frac{dZ}{Z} = \pi_1 \frac{dX_1}{X_1} + \ldots + \pi_n \frac{dX_n}{X_n}. \tag{15}$$

It follows from (15) that a balanced portfolio is a lognormal process and consequently has constants α_z and σ_z associated with it as in (1). Markowitz [1952, 1956, 1959] showed that a utility maximizing investor, with a concave utility function, would hold only those balanced portfolios which had a minimum σ_z for any given value of α_z. Such portfolios were called *efficient* and the CAPM was developed assuming that all investors held efficient portfolios.

One of the major results of the continuous time CAPM was that under unrestricted borrowing, all investors would hold a combination of a given balanced portfolio—the same portfolio for everyone—and cash. This balanced portfolio would, perforce, be equivalent to the market portfolio (excluding cash), since the sum of all investors' holdings must equal the market and each investor held the same relative proportions of stocks. Rosenberg and Ohlson [1976] observed that if the market portfolio were balanced, then all stocks would be equivalent, an obvious absurdity.

We can now see that a fundamental flaw of the continuous time CAPM was the failure to distinguish between passive portfolios and balanced portfolios. When viewed dynamically, the values of the proportions, A_I, at a given instant do not give a complete description of the state of the portfolio at that instant. It is necessary to distinguish between a balanced portfolio and a passive portfolio with which it happens to coincide momentarily. Apparently, this fact had been noticed before the creation of the continuous time CAPM. Latane [1963] observed that the multiple period performance of simple balanced portfolios usually differed from that of similar passive portfolios. Such analysis can be carried out more easily in the continuous time setting where stochastic integration can be applied, and we shall do so in the next section.

IV. Portfolio Dynamics and Excess Growth

In this section we investigate portfolio performance in a dynamic setting. As noted above, we do not consider questions dealing with optimal strategies for investors, and specifically avoid all normative issues.

In Section II we showed that the growth rate of a stock is a natural indicator of the stock's long term performance. Since a balanced portfolio is lognormal and hence has an associated growth rate, we shall first consider such a portfolio. Let us now analyze the case of cash and one stock because this simple example illustrates all the ideas that we wish to develop.

Consider a balanced cash-stock portfolio

$$Z = \pi Z + (1 - \pi)Z \tag{16}$$

where πZ is invested in a stock X with rate of return α and variance σ as in (1), and $(1 - \pi)Z$ is in $\$$. We assume that π is constant. The dynamics of Z are given by

$$dZ = \pi\frac{ZdX}{X} + (1 - \pi)\frac{Zd\$}{\$} \tag{17}$$

so,

$$\frac{dZ}{Z} = [r + \pi(\alpha - r]dt + \pi\sigma dB, \tag{18}$$

where r is the riskless rate of return. For the portfolio Z we have $\alpha_Z = r + \pi(\alpha - r)$ as rate of return, and $\sigma_Z{}^2 = \pi^2\sigma^2$ as variance.

For simplicity we shall assume that the stock has growth rate equal to the riskless rate r. Under this assumption we have $\alpha = r + \sigma^2/2$ so

$$\alpha_Z = r + \pi\frac{\sigma^2}{2}. \tag{19}$$

The growth rate for Z will be

$$\gamma_Z = \alpha_Z - \frac{\pi^2\sigma^2}{2} \tag{20}$$

so we have

$$\gamma_Z = r + (\pi - \pi^2)\frac{\sigma^2}{2}. \tag{21}$$

The term $(\pi - \pi^2)\sigma^2/2$ we shall call the *excess growth rate*, $\gamma_Z{}^*$. Notice that for $0 < \pi < 1$ the excess growth rate is positive and we see that the corresponding balanced portfolio will have a growth rate which is greater than that of either of its component stocks. For π outside the interval [0, 1] the excess growth rate is negative. The growth rate γ_Z will attain a maximum for $\pi = ½$, in which case $\gamma_Z = r + \sigma^2/8$ with excess growth rate $\gamma_Z{}^* = \sigma^2/8$.

The stochastic differential equation (17) can be integrated to give

$$\frac{Z(t)}{[X^{\pi}(t)\$^{(1 - \pi)}(t)]} = c\ \exp\left[\frac{(\pi - \pi^2)\sigma^2 t}{2}\right] \tag{22}$$

where c is constant. We see from (22) that the excess growth augments Z nonstochastically. The excess growth causes an exponential increase in the value of Z over the power weighted product of X and $\$$, and this increase is independent of the sample path of X.

Consider the following simple comparison. Let Z be a balanced cash-stock portfolio with $\pi = ½$, and let W be an passive cash-stock portfolio which starts with equal proportions of cash and stock. From equation (22) we see that every time that W returns to equal proportions——and this will occur infinitely often with probability one——Z will be greater than W by a factor of $\exp(\sigma^2 t/8)$. So Z indeed accrues excess growth, and in the concrete form of increased revenues.

For $0 < \pi < 1$, a balanced cash-stock portfolio will buy on a downstick and sell on an upstick. The act of rebalancing the portfolio is like an infinitesimal version of buying at the lows and selling at the highs. The continuous sequence of fluctuations in the price of the stock produces a constant accrual of revenues to the portfolio. This accrual of revenues is the excess growth. Such an accrual of revenues will, of course, be absent in a passive portfolio, so we would expect such a portfolio to lack excess growth.

Consider a passive portfolio

$$Z = v_1 X_1 + \dots + v_n X_n. \tag{23}$$

Since a passive portfolio is not usually lognormal, we need a more general definition of growth rate. Let $Z(t)$ be a real valued stochastic process. Then $Z(t)$ has *growth rate* γ if for any $\varepsilon > 0$

$$\exp(-\varepsilon t) < Z(t)\exp(-\gamma t) < \exp(\varepsilon t) \tag{24}$$

for all t greater than some finite value t_0 with probability one.

This definition coincides with the earlier definition when Z is a lognormal process. Now suppose that $X_1,\dots,X_n$ all have *growth rate* $\gamma = r$. It follows immediately from definition (24) that Z in (23) will also have growth rate r. Hence we see that a passive portfolio does not enjoy the excess growth which occurs in balanced portfolios.

To proceed further we need a precise mathematical definition of excess growth for a general class of portfolios. The term which represents the excess growth of Z in equation (22) occurs because the Ito calculus does not follow the rules of ordinary calculus, and when stochastic differentials are manipulated, Ito's lemma must be applied. Now it happens that another stochastic integral, the Stratonovich integral, is defined in such a way that it obeys the rules of ordinary calculus (see Stratonovich [1966]). This provides motivation to define the excess growth as the difference between the usual Ito version of a portfolio and the value it would assume if it were represented by a Stratonovich integral. Let us computate the value of this difference.

Suppose that we have stocks $X_1,\dots,X$ and cash and let X_i be defined by

$$dX_i = \alpha_i X_i dt + \sigma_i X_i dB_i \qquad i = 1 \dots n. \tag{25}$$

as in equation (1), but such that each stock has its own parameters α and σ and its own Brownian motion B. The correlation parameter between B_i and B_j is ρ_{ij}, and the covariance parameter between X_i and X_j is $\sigma_{ij} = \sigma_i\sigma_j\rho_{ij}$. We need not assume here that the growth rates $\gamma_i = \alpha_i - \sigma^2_i/2$ are equal.

Consider a portfolio Z with proportions $A_1,\dots,A_n$ of stocks $X_1,\dots,X_n$ respectively, and proportion $A_n = 1 - A_1 - \dots - A_n$ of cash. The proportions are functions $A_i = A_i(t, X_1(t),\dots,X_n(t))$ and we assume that they are continuously differentiable in each variable. In this case the difference between the Ito and Stratonovich integrals provides us with the definition of the *excess growth rate*, γ_Z^* of Z as

$$\gamma_Z^* = \frac{1}{2}\left(\sum_i A_i\sigma_i^2 - \sum_{ij}(D_jA_i + A_iA_j)\sigma_{ij}\right) \tag{26}$$

where D represents the partial derivative with respect to $\log X_i$. For a balanced portfolio, equation (26) reduces to

$$\gamma_Z^* = \frac{1}{2}\left(\sum_i \pi_i\sigma_i^2 - \sum_{ij}\pi_i\pi_j\sigma_{ij}\right), \tag{27}$$

since the partial derivatives vanish. It is not difficult to prove that for a balanced portfolio, $\gamma_Z^* \geq 0$ if $0 \leq \pi_i \leq 1$ for all $i = 0, 1, \dots, n$.

Consider a passive portfolio Z defined as in equation (23). Since the v_i are constant, the Ito and Stratonovich versions of Z will coincide, so we have zero excess growth. This can be confirmed by applying equation (26) with $A_i = v_iX_i/Z$ to show that $\gamma^*_Z = 0$.

Since the market portfolio is passive, it too will have zero excess growth. This fact is the basis for a stock market equilibrium model which we shall develop in the next section.

V. Stock Market Equilibrium and the Conservation of Excess Growth

The equilibrium model that we introduce here is a short term model in the sense that it governs the interactive behavior of the portfolios in the market at each instant in time. Despite the short term nature of the model, the equilibrium constraints it provides are nevertheless in terms of excess growth, which we have seen to be an indicator of long term performance. We make no assumptions about investor utility, and the resulting equilibrium resembles thermodynamic equilibrium more than the normative equilibrium of the CAPM.

Let the market be composed of stocks $X_1,\ldots,X_n$ and cash and let Z be a portfolio with continuously differentiable proportions $A_1,\ldots,A_n$. The n-tuple of proportions $A_1,\ldots,A_n$ parametizes Z up to equivalence since $A_0 = 1 - A_1 - \ldots - A_n$. The space of vector valued functions $A = (A_1,\ldots, A_n)$ therefore forms a parameter space Ω for the class of portfolios in the model. On the space Ω let $Z(dA)$ measure the total dollar value of those portfolios whose parameters fall within the volume element $dA \subset \Omega$. This measure will itself be a function of time and the stock prices.

Consider $A_i Z(dA)$, the amount invested in X_i in those portfolios whose parameters fall within dA. If we sum this quantity over the whole parameter space we get the value of X

$$X_i = \int A_i Z(dA) \tag{28}$$

where the integral represents summation over Ω. Let D_j be the partial derivative with respect to log X_j and note that $D_j Z = A_j Z$. We can differentiate (28) to get:

$$\int (D_j A_i - A_i A_j)\, Z(dA) = 0, \qquad i \neq j,$$

$$\int (A_i - D_i A_i - A_i^2)\, Z(dA) = 0. \tag{29}$$

These equations represent the *conservation of excess growth* in the market. The equations are valid for a wide class of stock price processes, but if we restrict ourselves to lognormal processes, there is a direct interpretation in terms of the excess growth we considered in the last section.

From equation (26) it follows that the total excess growth generated by all the portfolios in the market is

$$\int \gamma_Z^* Z(dA) = \int \tfrac{1}{2}\left(\textstyle\sum_i A_i \sigma_i^2 - \sum_{ij} (D_j A_i + A_i A_j)\sigma_{ij}\right) Z(dA). \tag{30}$$

Application of equations (29) shows that the right hand side of this equation vanishes which means that the total excess growth in the market is zero.

Suppose now that we restrict ourselves to balanced portfolios. In this case all portfolios have constant proportions $\pi_1,\ldots, \pi_n$ of stocks and $\pi_n = 1 - \pi_1 - \ldots - \pi_n$ of cash. The parameter space reduces to $\mathbf{R}^n$ n-dimensional real euclidean space, with parameter vectors $\pi = (\pi_1,\ldots,\pi_n)$. As above, we have a positive measure $Z(d\pi)$ on $\mathbf{R}^n$, and if we divide this measure by the total value M of the market, we get a probability distribution $P(d\pi) = Z(d\pi)/M$ which measures the dollar weighted proportion of the market which is held by those portfolios whose parameters lie within $d\pi$ Equation (28) becomes

$$\int \pi_i P(d\pi) = \Pi_i \tag{31}$$

where the integral is over $\mathbf{R}^n$ and $\Pi_i = X_i/M$, the global proportion of X_i in the market. Therefore the mean vector of the probability distribution P is $\Pi_1,\ldots,\Pi_n$. The conservation equations (29) become

$$\int \pi_i \pi_j P(d\pi) = 0, \qquad i \neq j,$$

$$\int (\pi_i - \pi_i^2) P(d\pi) = 0 \tag{32}$$

which, in terms of the covariance of P can be written

$$Var_p(\pi_i) = \Pi_i - \Pi_i^2,$$

$$Cov_p(\pi_i \pi_j) = -\Pi_i \Pi_j, \qquad i \neq j. \tag{33}$$

For any model which assumes that all investors hold balanced portfolios, equations (33) must be satisfied in order to avoid internal inconsistencies.

Now let us return to the continuous time CAPM. In that model all investors were assumed to hold the same relative proportions of all stocks, with only the relative proportion of cash differing from one portfolio to another. Therefore $\pi_i/\pi_j = \Pi_i/\Pi_j$, for all portfolios and the distribution P was concentrated on a line in $\mathbf{R}^n$. The only way in which the conservation equations (33) could be satisfied under these conditions was for the dimension of the parameter space to reduce to one, with all stocks equivalent, and this resulted in the degeneracy observed by Rosenberg and Ohlson [1976].

Although any consistent distribution P of portfolios must satisfy equations (33), these constraints do not uniquely determine P. In situations of this nature, it is customary to select that distribution which assumes the least additional information. The distribution assuming the least additional information is the maximum entropy distribution (see Levine and Tribus [1979]), which in this case is the multivariate normal with mean and variance given by equations (31) and (33) respectively. This distribution would imply that the greatest density of portfolios have proportions close to those of the market, but that there will also be some portfolios very unlike the market portfolio. This observation does not seem too unrealistic, but nevertheless the model is too restrictive.

An ample theory of stock market equilibrium must include portfolios with variable proportions to generate stock price movements. Perhaps more general stock price processes should be considered. But regardless of the level of generality, any consistent model must satisfy some version of the conservation equations (29).

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