An example of short-term relative arbitrage

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Abstract

Long-term relative arbitrage exists in markets where the excess growth rate of the market portfolio is bounded away from zero. Here it is shown that under a time-homogeneity hypothesis this condition will also imply the existence of relative arbitrage over arbitrarily short intervals.

Suppose we have a market of stocks $X_1, \ldots, X_n$ represented by positive continuous semimartingales that satisfy

$$d \log X_i(t) = \gamma_i(t) \, dt + \sum_{\nu=1}^d \xi_{i\nu}(t) \, dW_{\nu}(t),$$

for $i = 1, \ldots, n$, where $d \geq n \geq 2$, $(W_1, \ldots, W_d)$ is a $d$-dimensional Brownian motion, and the processes $\gamma_i$ and $\xi_{i\nu}$ are progressively measurable with respect to the underlying filtration with $\gamma_i$ locally integrable and $\xi_{i\nu}$ locally square-integrable. The process $X_i$ represents the total capitalization of the $i$th company, so the total capitalization of the market is $X(t) = X_1(t) + \cdots + X_n(t)$ and the market weight processes $\mu_i$ are defined by $\mu_i(t) = X_i(t)/X(t)$, for $i = 1, \ldots, n$. The $ij$th covariance process $\sigma_{ij}$ is defined by

$$\sigma_{ij}(t) \triangleq \sum_{\nu=1}^d \xi_{i\nu}(t)\xi_{j\nu}(t),$$

for $i, j = 1, \ldots, n$.

A portfolio $\pi$ is defined by its weights $\pi_1, \ldots, \pi_n$, which are bounded processes that are progressively measurable with respect to the Brownian filtration and add up to one. The portfolio value process $Z_\pi$ for $\pi$ satisfies

$$d \log Z_\pi(t) = \sum_{i=1}^n \pi_i(t) \, d \log X_i(t) + \gamma^*_{\pi}(t) \, dt, \quad \text{a.s.,}$$

where the process $\gamma^*_{\pi}$ defined by

$$\gamma^*_{\pi}(t) \triangleq \frac{1}{2} \left( \sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t) \right)$$

is called the excess growth rate process for $\pi$. It can be shown that if $\pi_i(t) \geq 0$, for $i = 1, \ldots, n$, then $\gamma^*_{\pi}(t) \geq 0$, a.s. The market weights $\mu_i$ define the market portfolio $\mu$, and if the market portfolio value process $Z_\mu$ is initialized so that $Z_\mu(0) = X(0)$, then $Z_\mu(t) = X(t)$ for all $t \geq 0$, a.s. Since the market weights are all positive, $\gamma^*_{\mu}(t) \geq 0$, a.s. This introductory material can be found in Fernholz (2002).

Let $S$ be the entropy function defined by

$$S(x) = -\sum_{i=1}^n x_i \log x_i,$$

for $x \in \Delta^n$, the unit simplex in $\mathbb{R}^n$. We see that $0 \leq S(x) \leq \log n$, where the minimum value occurs only at the corners of the simplex, and the maximum value occurs only at the point where $x_i = 1/n$ for all $i$. For a constant $c \geq 0$, the generalized entropy function $S_c$ is defined by

$$S_c(x) = S(x) + c,$$

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for $x \in \Delta^n$. It can be shown that $S_c$ generates a portfolio $\pi$ with weights
\[
\pi_i(t) = \frac{e^{-\log \mu_i(t)}}{S_c(\mu(t))} \mu_i(t),
\]
for $i = 1, \ldots, n$, and the portfolio value process $Z_\pi$ will satisfy
\[
d \log \left( \frac{Z_\pi(t)}{Z_\mu(t)} \right) = d \log S_c(\mu(t)) + \frac{\gamma^*_\mu(t)}{S_c(\mu(t))} dt, \quad \text{a.s.}
\] (1)
(see Fernholz (1999), Fernholz (2002), and Fernholz and Karatzas (2005)).

**Definition 1.** For $T > 0$, there is relative arbitrage versus the market on $[0, T]$ if there exists a portfolio $\pi$ such that
\[
P \left[ Z_\pi(T)/Z_\mu(T) \geq Z_\pi(0)/Z_\mu(0) \right] = 1,
\]
\[
P \left[ Z_\pi(T)/Z_\mu(T) > Z_\pi(0)/Z_\mu(0) \right] > 0.
\]
If $P \left[ Z_\pi(T)/Z_\mu(T) > Z_\pi(0)/Z_\mu(0) \right] = 1$, then this relative arbitrage is strong.

**Proposition 1.** For $T > 0$, suppose that for the market $X_1, \ldots, X_n$ there exists a constant $\varepsilon > 0$ such that
\[
\gamma^*_\mu(t) > \varepsilon, \quad \text{a.s.,}
\]
for all $t \in [0, T]$, and for the entropy function $S$
\[
\text{ess inf} \{ S(\mu(t)) : t \in [0, T/2] \} \leq \text{ess inf} \{ S(\mu(t)) : t \in [T/2, T] \}.
\] (2)
Then there is relative arbitrage versus the market on $[0, T]$.

**Proof.** Let
\[
A = \text{ess inf} \{ S(\mu(t)) : t \in [0, T/2] \}.
\] (3)
Since $\gamma^*_\mu(t) \geq \varepsilon > 0$ on $[0, T]$, a.s., not all the $\mu_i$ can be constantly equal to $1/n$, so
\[
0 \leq A < \log n.
\]
Hence, we can choose $\delta > 0$ such that $A + 2\delta < \log n$ and
\[
P \left[ \inf_{t \in [0, T/2]} S(\mu(t)) < A + \delta \right] > 0,
\]
so if we define the stopping time
\[
\tau_1 = \inf \{ t \in [0, T/2] : S(\mu(t)) \leq A + \delta \} \wedge T,
\]
then
\[
P \left[ \tau_1 \leq T/2 \right] > 0.
\]
We can now define a second stopping time
\[
\tau_2 = \inf \{ t \in [\tau_1, T] : S(\mu(t)) = A + 2\delta \} \wedge T,
\]
and we have $\tau_1 \leq \tau_2$, a.s.

Now consider the generalized entropy function
\[
S_\delta(x) \triangleq S(x) + \delta,
\]
for the same $\delta > 0$ as we chose above, so $S_\delta(x) \geq \delta$. It follows from (1) that
\[
\log \left( \frac{Z_\pi(\tau_2)/Z_\mu(\tau_2)}{Z_\pi(\tau_1)/Z_\mu(\tau_1)} \right) - \log \left( \frac{Z_\pi(\tau_1)/Z_\mu(\tau_1)}{Z_\pi(\tau_2)/Z_\mu(\tau_2)} \right) = \log S_\delta(\mu(\tau_2)) - \log S_\delta(\mu(\tau_1)) + \int_{\tau_1}^{\tau_2} \frac{\gamma^*_\mu(t)}{S_\delta(\mu(t))} dt, \quad \text{a.s.,}
\] (4)
for the times $\tau_1$ and $\tau_2$. Suppose we are on the set where $\tau_1 \leq T/2$, so $\tau_1 < \tau_2$, a.s., and consider two cases:
1. If \( \tau_2 < T \), then

\[
\log S_\delta(\mu(\tau_2)) - \log S_\delta(\mu(\tau_1)) \geq \log(A + 3\delta) - \log(A + 2\delta) > 0, \quad \text{a.s.}
\]

and since the integral in (4) is positive, a.s., we have

\[
\log \left( \frac{Z_\pi(\tau_2)}{Z_\mu(\tau_2)} \right) - \log \left( \frac{Z_\pi(\tau_1)}{Z_\mu(\tau_1)} \right) > 0, \quad \text{a.s.}
\] (5)

2. If \( \tau_2 = T \), then \( A + \delta < S_\delta(\mu(t)) < A + 3\delta \) for \( t \in [\tau_1, T] \), a.s., so

\[
\log S_\delta(\mu(\tau_2)) - \log S_\delta(\mu(\tau_1)) + \int_{\tau_1}^{\tau_2} \frac{\gamma_\mu^*(t)}{S_\delta(\mu(t))} \, dt > \log \frac{A + \delta}{A + 2\delta} + \frac{\varepsilon T}{2(A + 3\delta)}, \quad \text{a.s.}
\] (6)

Again there are two cases:

(a) If \( A = 0 \), let

\[
\delta = \frac{\varepsilon T}{6 \log 2},
\] (7)

so the left-hand side of the inequality in (6) will be positive, a.s., and (4) implies that

\[
\log \left( \frac{Z_\pi(\tau_2)}{Z_\mu(\tau_2)} \right) - \log \left( \frac{Z_\pi(\tau_1)}{Z_\mu(\tau_1)} \right) > 0, \quad \text{a.s.}
\] (8)

(b) If \( A > 0 \), then

\[
\lim_{\delta \downarrow 0} \left[ \log \frac{A + \delta}{A + 2\delta} + \frac{\varepsilon T}{2(A + 3\delta)} \right] = \frac{\varepsilon T}{2A} > 0,
\] (9)

so for small enough \( \delta > 0 \), (6) will be positive, and (8) will be valid.

Now consider the portfolio \( \eta \) defined by:

1. For \( t \in [0, \tau_1) \), \( \eta(t) = \mu(t) \), the market portfolio.

2. For \( t \in [\tau_1, \tau_2) \), \( \eta(t) = \pi(t) \), the portfolio generated by \( S_\delta \) with \( \delta \) chosen according to (7) or (9), as the case may be.

3. For \( t \in [\tau_2, T] \), \( \eta(t) = \mu(t) \).

If \( \tau_1 = T \), then \( \eta(t) = \mu(t) \) for all \( t \in [0, T] \), so

\[
\log \left( \frac{Z_\eta(T)}{Z_\mu(T)} \right) = \log \left( \frac{Z_\eta(0)}{Z_\mu(0)} \right), \quad \text{a.s.}
\]

If \( \tau_1 \neq T \), then \( \tau_1 \leq T/2 \) and \( \tau_1 < \tau_2 \), a.s. By the construction of \( \eta \), we have

\[
\log \left( \frac{Z_\eta(T)}{Z_\mu(T)} \right) - \log \left( \frac{Z_\eta(0)}{Z_\mu(0)} \right) = \log \left( \frac{Z_\pi(\tau_2)}{Z_\mu(\tau_2)} \right) - \log \left( \frac{Z_\pi(\tau_1)}{Z_\mu(\tau_1)} \right)
\] > 0, \quad \text{a.s.,}

with the inequality following from (5) or (8), as the case may be. Since \( \mathbb{P}[\tau_1 \neq T] > 0 \),

\[
\mathbb{P}\left[ \log \left( \frac{Z_\eta(T)}{Z_\mu(T)} \right) \geq \log \left( \frac{Z_\eta(0)}{Z_\mu(0)} \right) \right] = 1,
\]

\[
\mathbb{P}\left[ \log \left( \frac{Z_\eta(T)}{Z_\mu(T)} \right) > \log \left( \frac{Z_\eta(0)}{Z_\mu(0)} \right) \right] > 0,
\]

so there is relative arbitrage versus the market on \([0, T]\).

Let us recall that the market is diverse over the interval \([0, T]\) if there exists a \( \delta > 0 \) such that

\[
\mu_i(t) < 1 - \delta, \quad \text{a.s.,}
\]
for $i = 1, \ldots, n$ and all $t \in [0, T]$ (see, e.g., Fernholz (2002)).

**Corollary 1.** Let $T > 0$ and suppose that the market is not diverse over $[0, T/2]$ and that $\gamma_\mu^*(t) > \varepsilon > 0$ for $t \in [0, T]$. Then there is relative arbitrage versus the market on $[0, T]$.

*Proof.* In this case $A = 0$ in (3).

*Remark 1.* Corollary 1 can be applied to volatility-stabilized markets, for which Banner and Fernholz (2008) have previously shown the existence of short-term strong relative arbitrage.

*Remark 2.* The condition (2) can be generalized to a function $A$ defined on $[0, T]$ by

$$A(t) = \text{ess inf}\{S(\mu(t))\}.$$  

If $A$ increases over any subinterval of $[0, T]$, then an argument similar to that of case 1 in Proposition 1 will establish relative arbitrage. Moreover, Johannes Ruf has pointed out that the proof of Proposition 1 can be extended to establish relative arbitrage in the case where $A$ is slowly (enough) decreasing on $[0, T]$. By means of a remarkable construction, Karatzas and Ruf (2015) have shown that short-term relative arbitrage does not exist for arbitrary $A$.

**References**


